Fixed Point Theorem

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The rules for LAST can be found in [3].

Theorem 1 *For all formulas A and all variables y (free or not in A) there is a term f such that for every variable x (free or not in A)*

$$x \in f \vdash A[f/y]$$

$$A[f/y] \vdash x \in f$$

The Proof of the Fixed Point Using MALL

[2] gives a proof of the fixed point theorem in the MALL fragment with existential quantifiers enriched with unrestricted comprehension and equality axioms. [1] also presents a proof of this theorem for LAST which is used here.

Proof. Take fresh variables u, v, w and define the following terms

$$s \equiv \{z : \exists u \exists v(z = \langle u, v\rangle \otimes A[\{w : \langle w, v\rangle \in v\}/y, u/x])\}$$

$$f \equiv \{w : \langle w, s\rangle \in s\}$$

$A[f/y] \vdash x \in f$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \langle x, s\rangle = \langle x, s\rangle}\,\text{ax} \qquad \dfrac{}{A[f/y] \vdash A[f/y]}\,\text{ax}}{A[f/y] \vdash \langle x, s\rangle = \langle x, s\rangle \otimes A[\{w : \langle w, s\rangle \in s\}/y]}\,\otimes R}{A[f/y] \vdash \exists v(\langle x, s\rangle = \langle x, v\rangle \otimes A[\{w : \langle w, v\rangle \in v\}/y])}\,\exists R_1}{A[f/y] \vdash \exists u \exists v(\langle x, s\rangle = \langle u, v\rangle \otimes A[\{w : \langle w, v\rangle \in v\}/y, u/x])}\,\exists R_1}{A[f/y] \vdash \langle x, s\rangle \in s}\,\in R}{A[f/y] \vdash x \in f}\,\in R$$

The first existence introduction can be considered to be on the formula $B[s/v]$ where B is

$$B \equiv \langle x, s\rangle = \langle x, v\rangle \otimes A[f/y]$$

Likewise the second is on a formula $C[x/u]$ where C is

$$C \equiv \exists v(\langle x, s\rangle = \langle u, v\rangle \otimes A[\{w : \langle w, v\rangle \in v\}/y, u/x])$$

Notice that the replacement of the x for u and vice versa cancel as desired to give $A[f/y]$ but still allow for the application of existential introduction with the required sequent as the result.

$x \in f \vdash A[f/y]$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A[f/y] \vdash A[f/y]}\text{ax}}{\langle x,s\rangle = \langle x,s\rangle, A[f/y] \vdash A[f/y]}\text{weak}}{\langle x,s\rangle = \langle x,s\rangle \otimes A[f/y] \vdash A[f/y]}\otimes L}{\exists v(\langle x,s\rangle = \langle x,v\rangle \otimes A[\{w : \langle w,v\rangle \in v\}/y]) \vdash A[f/y]}\exists L}{\exists u \exists v(\langle x,s\rangle = \langle u,v\rangle \otimes A[\{w : \langle w,v\rangle \in v\}/y, u/x]) \vdash A[f/y]}\exists L}{\langle x,s\rangle \in s \vdash A[f/y]}\in L}{x \in f \vdash A[f/y]}\in L
$$

Note the use of weakening in this proof. Shirahata is ambiguous about what exact logic he is using perhaps that is why the proof is slightly more complicated not using weakening. Our purpose here is to prove this in LAST however so we may as well assume that we can use weak.

The Nature of x and y The sources do not make it clear what x and y are so I have guessed. On the hypothesis that indeed what im saying is true (as I hope to have proved) it may be instructive to consider some special cases.

Firstly, what if neither x or y appear in A? Then we see from the proof that f becomes something like the set of all sets or empty set because (reasoning loosely outside of LAST)

$$s = \{z : \langle u,v\rangle = z \otimes A\}$$

Thus if A is true then this is the set of all pairs and $f = \{w : \langle w,s\rangle \in \{\langle x,y\rangle\}\}$ which is all w. If A is false then this becomes the empty set and $f = \{w : w \in \emptyset\} = \emptyset$.

Consider now when y is not free in A. We get that $s = \{z : z = \langle u,v\rangle \otimes A[u/x]\} \sim \{z : A[\pi_1(z)/x]\}$ thus $f = \{w : A[w/x]\}$ i.e. f is the set of pairs such that when I put the first variable in A for x I get something true.

Next if x is not free we can "simplify" to get $s = \{z : A[f[\pi_2(z)/s]/y]\}$ thus $f = \{w : \langle w,s\rangle \in s\} \sim \{w : A[f/y]\}$ (b/c s is the second component).

Thus can be put together to think of the s and f terms as $s = \{\langle u,v\rangle : A[f[v/s]/y, u/x]\}$ and therefore $f = \{w : A[f/y, w/x]\}$. Thus

$$x \in f \iff A[f/y, x/x] = A[f/y]$$

Derived Rules for ILAL

LAST is formulated in ILAL (intuitionistic linear affine logic). Moreover Teruih defines connectives and so we attempt to deduce the standard connective rules from these definitions here.

Recall the in LAST the tensor of two formulas is defined via an arbitrary (but fixed) closed formula Θ:

$$A \otimes B \equiv \forall x.((A \multimap B \multimap \Theta \in x) \multimap \Theta \in x)$$

The rules is ILAL are in particular intuitionistic and so there should only be one term on the right hand side. This makes them strictly weaker than the ones that are used in the proof in MALL but upon examination of said MALL proof we see that we never exploit this power and only in fact need the intuitionistc derived rules below.

Lemma 1 *Right introduction for tensor*

$$\dfrac{\Delta_1 \vdash A \qquad \Delta_2 \vdash B}{\Delta_1, \Delta_2 \vdash A \otimes B}\ R\otimes$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Delta_1 \vdash A\end{matrix} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\\ \Delta_2 \vdash B\end{matrix} \qquad \dfrac{}{\Theta \in x \vdash \Theta \in x}\ \mathrm{Ax}}{\Delta_2, B \multimap \Theta \in x \vdash \Theta \in x}\ \multimap L}{\Delta_1, \Delta_2, A \multimap B \multimap \Theta \in x \vdash \Theta \in x}\ \multimap L}{\Delta_1, \Delta_2 \vdash (A \multimap B \multimap \Theta \in x) \multimap \Theta \in x}\ \multimap R}{\dfrac{\Delta_1, \Delta_2 \vdash \forall x.((A \multimap B \multimap \Theta \in x) \multimap \Theta \in x)}{\Delta_1, \Delta_2 \vdash A \otimes B}}\ \forall R$$

Where $x \notin FV(\Delta_1) \cup FV(\Delta_2)$

Lemma 2 *Left introduction for tensor*

$$\dfrac{A, B \vdash C}{A \otimes B \vdash C}\ L\otimes$$

Proof. Let x be a fresh (not free in C) variable. Then note that $C[t_0/x] \equiv C$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\\ A, B \vdash C[t_0/x]\end{matrix}}{A, B \vdash t_0 \in \{x : C\}}\ \in R}{A \vdash B \multimap t_0 \in \{x : C\}}\ \multimap R}{\vdash A \multimap B \multimap t_0 \in \{x : C\}}\ \multimap R \qquad \dfrac{\dfrac{}{C[t_0/x] \vdash C}\ \mathrm{ax}}{t_0 \in \{x : C\} \vdash C}\ \in L}{(A \multimap (B \multimap t_0 \in \{x : C\})) \multimap t_0 \in \{x : C\} \vdash C}\ \multimap L}{\dfrac{\forall y((A \multimap (B \multimap t_0 \in y)) \multimap t_0 \in y) \vdash C}{A \otimes B \vdash C}}\ \forall L$$

Where the last for-all introduction is done on the formula

$$((A \multimap (B \multimap t_0 \in y)) \multimap t_0 \in y)[\{x : C\}/y]$$

Again the LAST existential is a defined quantifier

$$\exists y.A \equiv \forall x.(\forall y.(A \multimap \Theta \in x) \multimap \Theta \in x)$$

Lemma 3 *Right existential introduction rule*

$$\frac{\Gamma \vdash A[t/y]}{\Gamma \vdash \exists y.A}\ \exists R$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\pi \\ \vdots \\ \Gamma \vdash A[t/y]\end{matrix} \qquad \dfrac{}{\Theta \in x \vdash \Theta \in x}\ \text{Ax}}{\Gamma, A[t/y] \multimap \Theta \in x \vdash \Theta \in x}\ \multimap L}{\Gamma, \forall y.(A \multimap \Theta \in x) \vdash \Theta \in x}\ \forall L}{\Gamma \vdash \forall y.(A \multimap \Theta \in x) \multimap \Theta \in x}\ \multimap R}{\Gamma \vdash \forall x.(\forall y.(A \multimap \Theta \in x) \multimap \Theta \in x)}\ \forall R}{\Gamma \vdash \exists y.A}$$

Where $x \notin FV(\Gamma)$

Lemma 4 *Left existential introduction rule*

$$\frac{A[t/x] \vdash \Gamma}{\exists x.A \vdash \Gamma}\ \exists L$$

Proof. Take a fresh variable γ so in particular $\gamma \notin FV(\Gamma)$ so that $\Gamma[\Theta/\gamma] \equiv \Gamma$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\pi \\ \vdots \\ A[t/x] \vdash \Gamma[\Theta/\gamma]\end{matrix}}{A[t/x] \vdash \Theta \in \{\gamma : \Gamma\}}\ \in R}{\vdash A[t/x] \multimap \Theta \in \{\gamma : \Gamma\}}\ \multimap R}{\vdash \forall x.(A \multimap \Theta \in \{\gamma : \Gamma\})}\ \forall R \qquad \dfrac{\dfrac{}{\Gamma[\Theta/\gamma] \vdash \Gamma}\ \text{Ax}}{\Theta \in \{\gamma : \Gamma\} \vdash \Gamma}\ \in L}{\forall x.(A \multimap \Theta \in \{\gamma : \Gamma\}) \multimap \Theta \in \{\gamma : \Gamma\} \vdash \Gamma}\ \multimap L}{\forall y.(\forall x.(A \multimap \Theta \in y) \multimap \Theta \in y)}\ \forall L}{\exists x.B \vdash \Gamma}$$

Similar to the proof of the left tensor rule.

Lemma 5 *Axiom rule for equality*

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a \in x \vdash a \in x}\ Ax}{\vdash a \in x \multimap a \in x}\ \multimap R}{\vdash \forall x.(a \in x \multimap a \in x)}\ \forall R}{\vdash a = a}$$

These derived rules prove that the proof supplied in MALL is in fact valid in LAST as well and so we have proved the fixed point theorem for LAST.

Why "Fixed Point"

Maybe its as simple as considering a formula as a function of its free variables like so

$$A[x, -] : terms \to formulas; \quad y \mapsto A[y/x]$$

Then the formula cannot have a fixed point because its input will be a term but its output will be a formula. We can "include" terms into the space of formluas via the function

$$\iota : terms \hookrightarrow formulas; \quad s \mapsto t \in s$$

becuase this is in some sense the simplest formula in LAST containing the term s. We could then interpret the fixed point of of a formula A to be a term such that

$$A[x, y] = \iota(y)$$

(contrast with the traditional fixed point $f : X \to X$, $fx = x$). The idea is already becoming clear but we need to consider the notion of equality on formulas. We have defined equality of terms to be Leibniz equality and it is natural to think of two formulas as being equal in a logical system if they prove one another (logically equivalent, linear iff).

Thus the meaning of a fixed point of such a function (the formulas) becomes clear; it is a term f such that

$$A[f/x] \dashv\vdash t \in f$$

Or maybe it has something to do with lambda calculus, Lawveres fixed point theorem and the internal logic of a Cartesian closed category; its tbd.

References

[1] Erik Istre. "Normalized naive set theory." In: 2017.

[2] M. Shirahata. "Fixpoint Theorem in Linear Set Theory". 1999. URL: https://www.fbc.keio.ac.jp/~sirahata/Research/fixpoint.pdf.

[3] Kazushige Terui. "Light Affine Set Theory: A Naive Set Theory of Polynomial Time". In: *Studia Logica* 77.1 (June 2004), pp. 9–40. ISSN: 0039-3215. DOI: 10.1023/B:STUD.0000034183.33333.6f. URL: http://link.springer.com/10.1023/B:STUD.0000034183.33333.6f (visited on 10/16/2022).